PLANE PLASTIC STRAIN PROBLEMS FOR ANISOTROPIC MATERIALS

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A general expression for the stress function in the plastic region of an orthotropic body is presented. The so-called "generalized complex variables" are introduced. They are similar to those introduced by Lekhnitskii for elastic materials [1,2]. A condition for this function to be real is derived. Based on this condition inverse problems can be solved. As illustrative examples some particular cases of states of stress are investigated. A comparison is made between the elastic and plastic complex parameters. An attempt is also made to establish a connection between the plastic and elastic stress functions.

1. Consider a homogeneous ideally plastic orthotropic body. The coordinate axes x, y, z are directed along the principal directions of orthotropy. The stress components σ_x, σ_y, τ_{xy} satisfy the following equilibrium conditions:

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0, \qquad \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=0 \tag{1.1}$$

and the von Mises-Hill [3] plasticity condition for an orthotropic body

$$\frac{(\sigma_x-\sigma_y)^2}{1-c}+4\tau_{xy}^2=4T^2 \qquad (-\infty<c<1) \tag{1.2}$$

where c depends on the anisotropic material constants; T is the yield stress in shear relative to the x-, y-axes. For $c = 0$ we obtain a well-known von Mises plasticity condition for an isotropic body.

We introduce the stress function F as follows:

$$\sigma_x=2T\sqrt{1-c}\,\frac{\partial^2F}{\partial y^2}, \qquad \sigma_y=2T\sqrt{1-c}\,\frac{\partial^2F}{\partial x^2}, \quad \tau_{xy}=-2T\sqrt{1-c}\,\frac{\partial^2F}{\partial x\partial y} \tag{1.3}$$

The equilibrium conditions are thus satisfied identically, and for the determination of F there remains the following equation:

$$D_2 D_1 F \overline{D}_2 \overline{D}_1 F = 1 \qquad (1.4)$$

where

$$D_k = \frac{\partial}{\partial y} - \mu_k \frac{\partial}{\partial x} = (\overline{\mu}_k - \mu_k) \frac{\partial}{\partial \overline{z}_k}, \qquad \mu_k = \pm \sqrt{c} + i \sqrt{1-c}, \quad (k = 1, 2) \qquad (1.5)$$

and the "generalized complex variables" are determined by

$$z_k = x + \mu_k y, \qquad \overline{z}_k = x + \overline{\mu}_k y \qquad (k = 1, 2)$$

Since function F is real, (1.4) can be written as $D_2 D_1 \overline{F D}_2 \overline{D}_1 \overline{F} = 1$ of $|D_2 D_1 F| = 1$ and thus

$$D_2 D_1 F = \exp[-i\theta] \qquad (1.6)$$

where $\theta = \theta(z_1, z_2; \overline{z}_1, \overline{z}_2)$ is an arbitrary real function.

The general solution of (1.6) is given by

$$F = \frac{1}{(\overline{\mu}_1 - \mu_1)(\overline{\mu}_2 - \mu_2)} \int d\overline{z}_1 \int \exp[-i\theta(z_1, z_2; \overline{z}_1, \overline{z}_2)]\, d\overline{z}_2 + F_1(z_1) + F_2(z_2) + p(x^2 + y^2) \qquad (1.7)$$

Here and in the sequel the integration is performed from fixed values of $\overline{z}_1^{\circ}$, $\overline{z}_2^{\circ}$ to arbitrary values $\overline{z}_1$, $\overline{z}_2$.

$F_1(z_1)$ and $F_2(z_2)$ are arbitrary analytic functions of their arguments; p = const.

The following relationship exists between the function θ and the angle α formed by slip lines with the x-axis at each point of a plastic region:

$$\tan\theta = -\sqrt{1-c}\,\tan 2\left(\alpha \pm \frac{1}{4}\pi\right)$$

For the isotropic case $c = 0$.

2. From the physical considerations it follows that only real functions are admissible.

Theorem 1. The necessary and sufficient condition for a function $F(z_1, z_2; \overline{z}_1, \overline{z}_2)$ to be real is

$$\frac{\partial^2 e^{-i\theta}}{\partial z_1 \partial z_2} = \frac{\partial^2 e^{i\theta}}{\partial \overline{z}_1 \partial \overline{z}_2} \qquad (2.1)$$

Proof. Operating on both sides of the identity $F \equiv \overline{F}$ by

$$\frac{\partial^4}{\partial z_1 \partial z_2 \partial \bar{z}_1 \partial \bar{z}_2}$$

and taking into account (1.7), results in (2.1) being a necessary condition. To show that (2.1) is also a sufficient condition we notice that from (2.1) it follows that

$$\frac{\partial^4 F}{\partial z_1 \partial z_2 \partial \bar{z}_1 \partial \bar{z}_2} = \frac{\partial^4 \bar{F}}{\partial z_1 \partial z_2 \partial \bar{z}_1 \partial \bar{z}_2}$$

and thus

$$F - \bar{F} = \Phi_1(z_1) + \Phi_2(z_2) + \bar{\psi}_1(\bar{z}_1) + \bar{\psi}_2(\bar{z}_2)$$

where $\Phi_k(z_k)$ and $\psi_k(z_k)$ $(k = 1, 2)$ are analytic functions of their arguments. Because the function F is defined by Formula (1.7), within arbitrary analytic functions, these functions can be selected in such a way that $F \equiv \bar{F}$.

Equation (2.1) can be written in Cartesian coordinates as follows:

$$\cos\theta\left(\frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial y^2} - 2\sqrt{1-c}\,\frac{\partial\theta}{\partial x}\,\frac{\partial\theta}{\partial y}\right) + \\ + \sin\theta\left(-2\sqrt{1-c}\,\frac{\partial^2\theta}{\partial x\partial y} + \left(\frac{\partial\theta}{\partial y}\right)^2 - \left(\frac{\partial\theta}{\partial x}\right)^2\right) = 0 \qquad (2.2)$$

and in polar coordinates (r, ϕ)

$$(\cos\theta\cos 2\varphi - \sqrt{1-c}\sin\theta\sin 2\varphi)\left(\frac{\partial^2\theta}{\partial r^2} - \frac{1}{r^2}\,\frac{\partial^2\theta}{\partial\varphi^2} - \frac{1}{r}\,\frac{\partial\theta}{\partial r}\right) + \\ + (\cos\theta\sin 2\varphi + \sqrt{1-c}\sin\theta\cos 2\varphi)\left(\frac{2}{r^2}\,\frac{\partial\theta}{\partial\varphi} - \frac{2}{r}\,\frac{\partial^2\theta}{\partial r\partial\varphi}\right) + \\ + (\sin\theta\sin 2\varphi - \sqrt{1-c}\cos\theta\cos 2\varphi)\frac{2}{r}\,\frac{\partial\theta}{\partial r}\,\frac{\partial\theta}{\partial\varphi} + \\ + (\sin\theta\cos 2\varphi + \sqrt{1-c}\cos\theta\sin 2\varphi)\left[\frac{1}{r^2}\left(\frac{\partial\theta}{\partial\varphi}\right)^2 - \left(\frac{\partial\theta}{\partial r}\right)^2\right] = 0 \qquad (2.3)$$

Letting $c = 0$ in (2.2) and (2.3) we obtain the corresponding conditions for the isotropic case [4]. Some other conditions can be specified for which the function F is real, e.g. such as those proposed in [4].

Theorem 2. A particular solution of (2.2), $\theta = \theta(x, y)$, which does not contain arbitrary parameters, determines the stress components to within a constant hydrostatic pressure.

Proof. If $\theta = \theta(x, y)$ is selected then the real function F is determined by (1.7) within an additive term $p(x^2 + y^2)$ which corresponds to a uniform hydrostatic pressure p. For it is easy to see that $D_2 D_1[p(x^2 + y^2)] = 0$. The theorem is proved.

Note 1. If $\theta = \theta(x, y)$ represents a solution of (2.2) then

$\psi_1 = -\theta(-x, y)$ and $\psi_2 = -\theta(x, -y)$ are also solutions of (2.2).

Conclusion. If $\theta = \theta(x, y)$ satisfies (2.2) then $\psi = \theta(-x, -y)$ also satisfies (2.2).

3. The complex parameters $\mu_k (k = 1, 2)$ characterize the anisotropy of the material. It is of interest to compare the elastic constants μ_k in [1,2] with the parameters μ_k introduced in this work. In [1] it was demonstrated that μ_k cannot be real. The same is true for the case of plastic anisotropy. For if μ_k were real then from (1.5) it follows that $c \geqslant 1$, but, as it is known from [3], the parameter c varies in an open interval $(-\infty, 1)$. Hence the case $c \geqslant 1$ is void of any physical meaning.

Let now $\mu_k = a_k + i\beta_k$; a_k and β_k are real, then for $-\infty < c \geqslant 0$, $a_k = 0$, $\beta_k = \sqrt{(1-c)} \pm \sqrt{-c}$, i.e. the parameters μ_k are pure imaginary, and for $0 < c < 1$, $a_k = \pm\sqrt{c}$, $\beta_k = \sqrt{(1-c)}$, thus β_k is always positive.

Moreover, the equality $\mu_1 = \mu_2$ for the plastic case holds good only for isotropic materials. In the elastic case the equality of these parameters is possible for anisotropic materials as well.

In [1] some combinations of μ_1 and μ_2 are shown which are real for orthotropic cases. Below we present these combinations together with similar combinations for plastic regions.

	Elastic region	Plastic region
$\mu_1\mu_2$	$-\sqrt{\frac{\beta_{22}}{\beta_{11}}}$	-1
$-i(\mu_1+\mu_2)$	$\sqrt{\frac{2\beta_{12}+\beta_{66}}{\beta_{11}} + 2\sqrt{\frac{\beta_{22}}{\beta_{11}}}}$	$2\sqrt{1-c} = \frac{2XZ}{T\sqrt{4Z^2 - X^2}}$
$\mu_1^2+\mu_2^2$	$-\frac{2\beta_{12}+\beta_{66}}{\beta_{11}}$	$2(2c-1) = 2\left(1 - \frac{2X^2Z^2}{T^2(4Z^2 - X^2)}\right)$

Since in [1] the combinations of the complex parameters are given for the generalized plane-stress problem, the constants a_{ij} there must be replaced by β_{ij}. This is so because the generalized plane-stress problem is identical with the plane-strain problem if the constants a_{ij} in the former are replaced by β_{ij}, where

$$\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}} \qquad (i, j = 1, 2, 6)$$

$X = Y$, Z are the yield stresses in tension in the principal orthotropic directions, T is the yield stress in shear relative to x-, y-axes. The relationship $X = Y$ follows directly from the von Mises-Hill condition and from the assumption of the existence of a plastic potential for anisotropic materials [3].

Note 2. At the price of some additional calculational complexities, one can obtain the formulas analogous to (1.7) and (2.1) for more general cases or anisotropy.

4. Using (2.2) and (2.3) we obtain, now, several particular solutions of equilibrium of anisotropic bodies in the plastic region. It will be not necessary to determine the stress functions for each individual case.

If θ is known, then the stress components can easily be found from the equilibrium equations (1.1).

1) $\theta = \alpha = \text{const}$. This is the simplest solution corresponding to the uniform stress field

$$\sigma_x = \cos\alpha + p, \quad \sigma_y = p, \quad \tau_{xy} = \frac{1}{2\sqrt{1-c}}\sin\alpha \tag{4.1}$$

Here and in the sequel the stresses are taken relative to the quantity $2T\sqrt{(1-c)}$.

2) We seek now a solution in the form $\theta = \theta(y)$. From (2.2) we have

$$\frac{d^2\theta}{dy^2} - \tan\theta\left(\frac{d\theta}{dy}\right)^2 = 0 \quad \text{or} \quad \theta = \sin^{-1}(Ay + B)$$

The state of stress corresponding to the above relationships is realized in a strip

$$-\frac{1+B}{A} \leqslant y \leqslant \frac{1-B}{A}, \qquad A > 0$$

compressed by the rough plates.

The stress components are

$$\sigma_x = -\frac{A}{\sqrt{1-c}}x + \sqrt{1-(Ay+B)^2} + p \tag{4.2}$$

$$\sigma_y = -\frac{A}{\sqrt{1-c}}x + p, \qquad \tau_{xy} = \frac{1}{2\sqrt{1-c}}(Ay+B)$$

For $c = 0$ we obtain an analogous isotropic solution [4].

3) We seek now a solution in the form $\theta = \theta(\phi)$. Equations (2.4) will be transformed into the following:

$$(\cos\theta\cos 2\varphi - \sqrt{1-c}\sin\theta\sin 2\varphi)(-d^2\theta / d\varphi^2) +$$
$$+ (\cos\theta\sin 2\varphi + \sqrt{1-c}\sin\theta\cos 2\varphi)\, 2d\theta / d\varphi +$$
$$+ (\sin\theta\cos 2\varphi + \sqrt{1-c}\cos\theta\sin 2\varphi)(d\theta / d\varphi)^2 = 0$$

One possible solution of the above is

$$\theta = \tan^{-1} \frac{1}{\sqrt{1-c}\tan 2\varphi}$$

This stress field is realized in a plastic wedge loaded uniformly along its edges.

For $c \geqslant 0$ the stress components are

$$\sigma_r = \frac{-c\sin 2\varphi\cos 2\varphi}{\sqrt{(1-c)(1-c\sin^2 2\varphi)}} + \frac{1}{2\sqrt{1-c}} E(\sqrt{c};\varphi) + p \tag{4.3}$$

$$\sigma_\varphi = \frac{1}{2\sqrt{1-c}} E(\sqrt{c};\varphi) + p, \qquad \tau_{r\varphi} = -\frac{1}{2}\sqrt{\frac{1-c\sin^2 2\varphi}{1-c}}$$

For $c \leqslant 0$ the stresses are

$$\sigma_r = \frac{-c\sin 2\varphi\cos 2\varphi}{\sqrt{(1-c)(1-c\sin^2 2\varphi)}} - \frac{1}{2} E\left(\sqrt{\frac{c}{c-1}};\varphi\right) + p \tag{4.4}$$

$$\sigma_\varphi = -\frac{1}{2} E\left(\sqrt{\frac{c}{c-1}};\varphi\right) + p, \qquad \tau_{r\varphi} = -\frac{1}{2}\sqrt{\frac{1-c\sin^2 2\varphi}{1-c}}$$

where $E(k, \phi)$ is a normal form of the Legendre elliptical integral of the second kind. For the case of an isotropic wedge ($c = 0$) Formulas (4.3) and (4.4) are expressed as follows [5]:

$$\sigma_r = \sigma_\varphi = -\varphi + p, \qquad \tau_{r\varphi} = \frac{1}{2}$$

5. To solve elasto-plastic problems it is necessary to know the relationship between the plastic and elastic constants. If, for instance, at the elasto-plastic boundary the complex parameters suffer no jump (this seems to be quite a natural proposition, since, for the isotropic case, μ_1 and μ_2 are generally the same in plastic and elastic regions) then the following scheme may be proposed which would permit the continuity conditions to be satisfied over the elasto-plastic boundary [6]. Let the elastic solution be [2]

$$F^\circ = 2\,\mathrm{Re}\,[F_1^\circ(z_1) + F_2^\circ(z_2)] \tag{5.1}$$

then the plastic solution can be represented as

$$F = F_0(z_1, z_2; \bar{z}_1, \bar{z}_2) + \varkappa(z_1, z_2; \bar{z}_1, \bar{z}_2) \tag{5.2}$$

Moreover, we have

$$F_0 = 2\mathrm{Re}\,[F_1(z_1) + F_2(z_2)]$$

$$\varkappa = \frac{1}{(\bar{\mu}_1 - \mu_1)(\bar{\mu}_2 - \mu_2)} \int d\bar{z}_1 \int \exp\,[-i\theta(z_1, z_2; \bar{z}_1, \bar{z}_2)]\, d\bar{z}_2 - \overline{F_1(z_1)} - \overline{F_2(z_2)} + p(x^2 + y^2) \tag{5.3}$$

where $\kappa = \kappa(z_1, z_2; \bar{z}_1, \bar{z}_2)$ is a real function (cf. (1.7)).

Let us introduce the following notation [2]:

$$\frac{dF_k^\circ(z_k)}{dz_k} = \Phi_k^\circ(z_k), \qquad \frac{dF_k(z_k)}{dz_k} = \Phi_k \quad) \quad (k = 1, 2) \tag{5.4}$$

On the elasto-plastic boundary γ we require that the following inequalities be satisfied:

$$2\text{Re}\,[\Phi_1^\circ(z_1) + \Phi_2^\circ(z_2)]_\gamma = 2\text{Re}\,[\Phi_1(z_1) + \Phi_2(z_2)]_\gamma, \qquad \frac{\partial \varkappa}{\partial z_1}\Big|_\gamma = \frac{\partial \varkappa}{\partial z_2}\Big|_\gamma = 0$$
$$2\text{Re}\,[\mu_1\Phi_1^\circ(z_1) + \mu_2\Phi_2^\circ(z_2)]_\gamma = 2\text{Re}\,[\mu_1\Phi_1(z_1) + \mu_2\Phi_2(z_2)]_\gamma,$$

Thus, on γ the equilibrium conditions are satisfied, and the elastic stress function F° is continuously transformed into a plastic stress function F.

I would like to take this opportunity to draw attention to the work [4]. The question of the plastic stress functions being biharmonic was considered in [7,8].

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